Convective Heat Transfer (6)
Forced Convection (8)

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Agenda

• Convective heat transfer
• Continuity eq.
• Convective duct flow (introduction to ch. 8)
Convective heat transfer

FIG. 5-1
Sketch showing different boundary-layer flow regimes on a flat plate.

\[ \text{Re}_x = \frac{u_\infty x}{\nu} \]
Convective heat transfer

**FIG. 5-2**
Laminar velocity profile on a flat plate.
Convective heat transfer

\[ \dot{Q} = \alpha(t_f - t_w)A \]
Convective heat transfer

\[ \dot{Q} = \alpha(t_f - t_w)A \]

\[ x = 0 \Rightarrow u, v, w = 0 \Rightarrow \text{heat conduction in the fluid} \]

Introduction of heat transfer coefficient:

\[ \alpha = \frac{\mathcal{G}/A}{t_w - t_f} = -\frac{\lambda_f \left( \frac{\partial t}{\partial y} \right)_{x=0}}{t_w - t_f} \quad (6-3) \]
Convective heat transfer

Objective (of chapter 6-11):
Determine $\alpha$ and the parameters influencing it for prescribed $t_w(x)$ or $q_w(x) = Q/A$
**Order of magnitude for \( \alpha \)**

<table>
<thead>
<tr>
<th>Medium</th>
<th>( \alpha ) W/m(^2)K</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air (1 bar); natural convection</td>
<td>2-20</td>
</tr>
<tr>
<td>Air (1 bar); forced convection</td>
<td>10-200</td>
</tr>
<tr>
<td>Air (250 bar); forced convection</td>
<td>200-1000</td>
</tr>
<tr>
<td>Water; forced convection</td>
<td>500-5000</td>
</tr>
<tr>
<td>Organic liquids; forced convection</td>
<td>100-1000</td>
</tr>
<tr>
<td>Condensation (water)</td>
<td>2000-50000</td>
</tr>
<tr>
<td>Condensation (organic vapors)</td>
<td>500-10000</td>
</tr>
<tr>
<td>Evaporation, boiling, (water)</td>
<td>2000-100000</td>
</tr>
<tr>
<td>Evaporation, boiling (organic liquids)</td>
<td>500-50000</td>
</tr>
</tbody>
</table>
How to do it? (to describe convective HT)
What are the tools?

**Fluid motion:**
Mass conservation equation (Continuity eqn)
Momentum equations (Newton’s second law)

**Energy balance in the fluid**
First law of thermodynamics for an open system
Continuity eq.
Expresses that mass is constant and not destroyable.

\[
\frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0 \quad (6-4)
\]

Especially for
steady state, incompressible flow,
two-dimensional case

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (6-5)
\]
Resulting momentum equations – 2 dim.

\[ \hat{x} : \quad \rho \left( \frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \rho F_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]

\[ \hat{y} : \quad \rho \left( \frac{\partial v}{\partial \tau} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \rho F_y - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \]

Impossible to solve by hand, need to be simplified… (chapter 6, 7 and 8)
Temperature Equation

\[
\frac{\partial t}{\partial x} u + \frac{\partial t}{\partial y} v + \frac{\partial t}{\partial z} w = \frac{\lambda}{\rho c_p} \left( \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} \right)
\]

Unpossible to solve by hand, need to be simplified… (chapter 6, 7 and 8)
Boundary layer approximations (laminar case)

Why different fields for temperature and velocity ???
Boundary layer theory developed by Prandtl

\[ u \gg v \]

If the boundary layer thickness is very small

If 2D

\[ \frac{\partial u}{\partial y} \gg \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \]

\[ \frac{\partial t}{\partial y} \gg \frac{\partial t}{\partial x} \]
Boundary layer approximations – Prandtl’s theory

\[ p = p(x) \]

From Navier-Stokes eqn in the y-direction it is found that the pressure is independent of y

\[ \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \rho F_x - \frac{dp}{dx} + \mu \frac{\partial^2 u}{\partial y^2} \]

Then Navier Stokes in the x-direction is simplified to:

\[ u \frac{\partial t}{\partial x} + v \frac{\partial t}{\partial y} = \frac{\lambda}{\rho c_p} \frac{\partial^2 t}{\partial y^2} \]

Also the temperature field is simplified
Boundary layer approximations – Prandtl’s theory

\[ p + \frac{1}{2} \rho U^2 = \text{konstant} \]

Bernoulli’s eqn describes the flow outside the boundary layer

\[ \frac{dp}{dx} = -\rho U \frac{dU}{dx} \]

\[ \text{Pr} = \frac{\nu \rho c_p}{\lambda} = \frac{\mu c_p}{\lambda} \]

The dimensionless Prandtl number is introduced
Boundary layer equations

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

Mass conservation

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2} \]

Momentum conservation

\[ u \frac{\partial t}{\partial x} + v \frac{\partial t}{\partial y} = \frac{\mu}{\rho \text{Pr}} \frac{\partial^2 t}{\partial y^2} \]

Energy conservation
Boundary layers

\[ 5 \cdot 10^5 \]

\[ \text{Re}_c = U_\infty x_c / \nu \]

\[ \text{Nu} = f_7 (\text{Re}, \text{Pr}) \]
**Continuity eq.** (expresses that mass is constant and not destroyable)

\[ m_{x_1} = \rho u \, dy \, dz \]

\[ \Delta \dot{m}_x = \frac{\partial}{\partial x} (\rho u) dx \, dy \, dz \]

Net mass flow out in x-direction

Analogous in y- and z-directions

\[ \Delta \dot{m}_y = \frac{\partial}{\partial y} (\rho v) dy \, dx \, dz \]
\[ \Delta \dot{m}_z = \frac{\partial}{\partial z} (\rho w) dz \, dx \, dy \]

**Netto utströmmat**, net flow out : \( \Delta \dot{m}_x + \Delta \dot{m}_y + \Delta \dot{m}_z \)

Net mass flow out \( \Rightarrow \) Reduction in mass within volume element
Cont. continuity eq.

Reduction per time unit:

\[
\frac{\partial \rho}{\partial \tau} \, dx \, dy \, dz
\]

\[
\therefore \quad - \frac{\partial \rho}{\partial \tau} = \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w)
\]

Consider a mass balance for the volume element at the previous page

\[
\frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) = 0 \quad (6 - 4)
\]

Especially for steady state, incompressible flow, two-dimensional case

\[
\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (6 - 5)
\]
Navier – Stokes’ ekvationer (eqs.)
Derived from Newton’s second law

\[
m \cdot \vec{a} = \vec{F}
\]

\[
m = \rho \, dx \, dy \, dz
\]

\[
\vec{a} = \left( \frac{du}{d\tau}, \frac{dv}{d\tau}, \frac{dw}{d\tau} \right)
\]

but

\[
u = u(x, y, z, \tau), \quad v = v(x, y, z, \tau), \quad w = w(x, y, z, \tau)
\]
Forces

\[ \vec{F} \]  

The surface forces act on the boundary surfaces of the fluid element and are acting as either normal forces or shear forces.

a. volume forces \((F_x, F_y, F_z)\) are calculated per unit mass, \(N/kg\)

b. stresses \(\sigma_{ij}\) \(N/m^2\)
**Forces** The surface forces are calculated per unit area and are called stresses

\[ \vec{F} \]

a. volume forces \((F_x, F_y, F_z)\) are calculated per unit mass, \(\text{N/kg}\)

b. stresses \(\sigma_{ij}\) \(\text{N/m}^2\)

\[
\sigma_{ij} = \begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{bmatrix}
\]

Stresses for an element \(\text{dxdydz}\)
Signs for the stresses
Signs for the stresses

\[ \sigma_{xx} \]
\[ \sigma_{yy} \]
\[ \sigma_{yx} \]
\[ \sigma_{xy} \]

Resulting stresses

\[ \sigma_{xx} + \frac{\partial}{\partial x} (\sigma_{xx})dx \]
\[ \sigma_{xy} + \frac{\partial}{\partial x} (\sigma_{xy})dx \]
\[ \sigma_{yx} + \frac{\partial}{\partial y} (\sigma_{yx})dy \]
\[ \sigma_{yy} + \frac{\partial}{\partial y} (\sigma_{yy})dy \]
Net force in x-direction (for the 3D case)

\[ \frac{\partial}{\partial x}(\sigma_{xx})dx dy dz + \frac{\partial}{\partial y}(\sigma_{yx})dy dx dz + \frac{\partial}{\partial z}(\sigma_{zx})dz dx dy \]

\[ \frac{\partial}{\partial x_i}(\sigma_{ji})dx dy dz \]
Examples of stresses

\[ \sigma_{xx} = -p + 2\mu e_{xx} = -p + 2\mu \frac{\partial u}{\partial x} \]

\[ \sigma_{xy} = \sigma_{yx} = 2\mu e_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]

\[ \sigma_{yy} = -p + 2\mu e_{yy} = -p + 2\mu \frac{\partial v}{\partial y} \]
Resulting momentum equations

\[ \hat{x} : \rho \left( \frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \rho F_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]

\[ \hat{y} : \rho \left( \frac{\partial v}{\partial \tau} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \rho F_y - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \]
Energy eq. (First law of thermodynamics of an open system), ⇒ Temperature field eq.

\[ dQ = dH \]

Neglecting kinetic and potential energy

Net heat to element =

Change of enthalpy flow
Heat conduction in the fluid (calculating the heat flow as in chapter 1)

\[
\dot{Q}_x = -\lambda A \frac{\partial t}{\partial x} = -\lambda dydz \frac{\partial t}{\partial x}
\]

\[
\dot{Q}_{x+dx} = \dot{Q}_x + \frac{\partial \dot{Q}_x}{\partial x} \, dx =
\]

\[
= -\lambda \frac{\partial t}{\partial x} dydz - \frac{\partial}{\partial x} (\lambda \frac{\partial t}{\partial x}) dx dydz
\]

\[
\Delta \dot{Q}_x = \dot{Q}_{x+dx} - \dot{Q}_x = -\frac{\partial}{\partial x} (\lambda \frac{\partial t}{\partial x}) dx dydz
\]

6.24
Analogous in y- and z-directions

\[
\Delta \dot{Q}_y = - \frac{\partial}{\partial y} (\lambda \frac{\partial t}{\partial y}) dy \ dx \ dz
\]

\[
\Delta \dot{Q}_z = - \frac{\partial}{\partial z} (\lambda \frac{\partial t}{\partial z}) dz \ dx \ dy
\]

\[
d\dot{Q} = \Delta \dot{Q}_x + \Delta \dot{Q}_y + \Delta \dot{Q}_z
\]

\[
sign \text{ convention for heat } d\dot{Q}
\]

\[
d\dot{Q} = \left\{ \frac{\partial}{\partial x} (\lambda \frac{\partial t}{\partial x}) + \frac{\partial}{\partial y} (\lambda \frac{\partial t}{\partial y}) + \frac{\partial}{\partial z} (\lambda \frac{\partial t}{\partial z}) \right\} dx \ dy \ dz
\]

(6-27)
Enthalpy flows and changes

• Flow of enthalpy in the x-direction

\[ \dot{H}_x = \dot{m}_x h = \rho u \, dy \, dz \, h \]

\[ \Rightarrow \, d\dot{H}_x = \rho h \frac{\partial u}{\partial x} \, dx \, dy \, dz + \rho u \frac{\partial h}{\partial x} \, dx \, dy \, dz \]
Enthalpy changes

• y- and z-directions

\[
d\dot{H}_y = \rho h \frac{\partial v}{\partial y} dx\,dy\,dz + \rho v \frac{\partial h}{\partial y} dx\,dy\,dz \quad 6.29
\]

\[
d\dot{H}_z = \rho h \frac{\partial w}{\partial z} dx\,dy\,dz + \rho w \frac{\partial h}{\partial z} dx\,dy\,dz \quad 6.30
\]
Total change in enthalpy

\[ dH = dH_x + dH_y + dH_z = \]

\[ = \rho h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dx dy dz + \]

\[ \rho \left( u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} + w \frac{\partial h}{\partial z} \right) dx dy dz \]
Enthalpy vs temperature

\[ h = h(p, t) \]

\[ \Rightarrow dh = \left( \frac{\partial h}{\partial p} \right)_t dp + \left( \frac{\partial h}{\partial t} \right)_p dt \]

6.33
**Enthalpy vs temperature**

\[
c_p = \left( \frac{\partial h}{\partial t} \right)_p
\]

By definition

For ideal gases the enthalpy is independent of pressure, i.e.,
\[
(\partial h / \partial p)_t \equiv 0
\]

For liquids, one commonly assumes that the derivative
\[
(\partial h / \partial p)_t
\]
is small and/or that the pressure variation \(dp\) is small compared to the change in temperature.

Then generally one states \(dh = c_p \, dt\)

i.e., enthalphy is coupled to temperature via the heat capacity
Temperature Equation

Rewriting eqn 6.32 as a function of T instead of h

\[ u \frac{\partial t}{\partial x} + v \frac{\partial t}{\partial y} + w \frac{\partial t}{\partial z} = \frac{\lambda}{\rho c_p} \left( \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} \right) \]
Chapter 8 - Convective Duct Flow

FIG. 5-3
Velocity profile for (a) laminar flow in a tube and (b) turbulent tube flow.
Chapter 8 - Convective Duct Flow

\[ \dot{m} = \rho A u_m \]

\[ \text{Re}_D = \frac{u_m D}{v} \]

\[ \frac{u}{u_{\max}} = \left(1 - \frac{y^2}{b^2}\right) \]

Parallel plate duct

\[ \frac{u}{u_m} = 2 \left\{1 - \left(\frac{r}{R}\right)^2\right\} \]

Circular pipe, tube

Laminar if pipe or tube \( \text{Re}_D < 2300 \)
Chapter 8 Convective Duct Flow

\[
\frac{L_i}{D} = 0.0575 \text{Re}_D
\]
Cont. duct flow

If \( Re_D > 2300 \)

Laminar boundary layer  Turbulent boundary layer

Fully developed turbulent flow

Laminar boundary layer  Turbulent boundary layer

omslag, transition

fullt utbildad turbulent strömning
Pressure drop fully developed flow

\[ \Delta p = f \frac{L}{D_h} \frac{\rho u_m^2}{2} \]
\[ f = \frac{C}{Re} \]
\[ Re = \frac{u_m D_h}{\nu} \]

\[ D_h = \text{hydraulic diameter} \]

\[ D_h = \frac{4 \times \text{cross section area}}{\text{perimeter}} \]

\[ u_m = \frac{\dot{m}}{\rho A} \]
Pressure drop - entrance region (circular pipe)

Figure 8.4
Convective heat transfer for an isothermal tube

\[ t_w = \text{konstant; constant} \]
Convective heat transfer for an isothermal tube

Velocity field fully developed

\[ u = 2u_m \left( 1 - \left( \frac{r}{R} \right)^2 \right) \]

"Heat balance" for an element \( \text{dxdr} 2\pi r \)

Heat conduction in radial direction

Enthalpy transport in x-direction
Boundary conditions:

\[ x = 0 : t = t_0 \]
\[ r = R : t = t_w \]
\[ r = 0 : \frac{\partial t}{\partial r} = 0 \text{ (Symmetry)} \]

Remember the figure from previous page

Energy eqn for steady state

\[ \rho c_p u \frac{\partial t}{\partial x} = \lambda \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial t}{\partial r} \right) \]

"(8 – 12)"

Commonly called Graetz problem
Introduce \( r' = r|R, \quad x' = x|R, \quad \Theta = t - t_w, \)

\[
a = \frac{\lambda}{\rho c_p}
\]

Nusselt approach still valid

\[
\Rightarrow \quad \frac{u}{a} \frac{1}{R} \frac{\partial \Theta}{\partial x'} = \frac{1}{R r'} \frac{\partial}{\partial r'} \left( R r' \frac{\partial \Theta}{R \partial r'} \right)
\]

Introduce \( u = 2u_m (1 - r'^2) \quad \text{Coupling between velocity and average velocity} \)

\[
\frac{2u_m R}{a} \frac{\partial \Theta}{\partial x'} = \frac{1}{r'(1 - r'^2)} \frac{\partial}{\partial r'} \left( r' \frac{\partial \Theta}{\partial r'} \right)
\]

or

\[
\text{Re}_D \text{ Pr} \frac{\partial \Theta}{\partial x'} = \frac{1}{r'(1 - r'^2)} \frac{\partial}{\partial r'} \left( r' \frac{\partial \Theta}{\partial r'} \right) \quad (8-20)
\]
Steady heat conduction

\[ \text{Re}_D \text{ Pr} \frac{\partial \Theta}{\partial x'} = \frac{1}{r'(1 - r'^2)} \frac{\partial}{\partial r'} \left( r' \frac{\partial \Theta}{\partial r'} \right) \] 

(8 - 20)

compared with unsteady heat conduction:

\[ \frac{\partial t}{\partial \tau} = a \left\{ \frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \frac{\partial t}{\partial r} \right\} = a \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial t}{\partial r} \right) \]
Assume \( \vartheta = F(x') \cdot G(r') \).

After some (8.24-8.28) calculations one finds

\[
\vartheta = \sum_{i=0}^{\infty} C_i G_i (r') e^{-\beta_i^2 x'/Re_D Pr} \quad (8 - 29)
\]

\( \beta_0 < \beta_1 < \beta_2 < \beta_3 < \beta_4 \ldots \)
Temperature profile in the thermal entrance region for a circular pipe with uniform wall temperature and fully developed laminar flow

\[
\begin{align*}
\beta_0 & = 2.705 \\
\beta_1 & = 6.667 \\
\beta_2 & = 10.67
\end{align*}
\]
the enthalpy flow of the mixture: \( \dot{m}c_p t_B \)

the enthalpy flow can be written

\[
\int_0^R \rho 2\pi r dr u c_p t \Delta \bar{m} \text{ "h"} \]

\[
\therefore t_B = \frac{1}{\bar{m}} \rho 2\pi \int_0^R ur t dr
\]

\[
\dot{m} = \int_0^R \rho 2\pi r dr u \left( \frac{\pi D^2}{4} \right) u_m
\]

\[
t_B = \frac{\int_0^R ur t dr}{\int_0^R ur dr}
\]

(8-34)
Local Nusselt number

- Local Nusselt number
- konstant värmeflöde, uniform heat flux
- konstant temperatur, constant wall temperature
- Hastighetsfält ej fullt utbildat; velocity field not fully developed

Graph showing the relationship between $N_{ud}$ and $x/D$ for different conditions.


**Average Heat Transfer Coefficient**

1. If the velocity field is fully developed ⇒

   \[
   \overline{Nu}_D = \frac{\bar{\alpha}D}{\lambda} = \left\{3.656 + \frac{0.0668 \text{Re}_D \text{Pr} D}{x} \right\} \left(\frac{\mu_B}{\mu_w}\right)^{0.14}
   \]

   N.B.! Higher values if velocity field not fully developed. Eq. (8-38) gives the average value

   \[
   \overline{Nu}_D = \frac{\bar{\alpha}D}{\lambda} = 1.86 \left\{\frac{\text{Re}_D \text{Pr}}{L/D}\right\}^{1/3} \left(\frac{\mu_B}{\mu_w}\right)^{0.14}
   \]

   \[
   L/D < 0.1 \quad \text{Re}_D \text{Pr} < 0.1 \quad t_w = \text{constant}
   \]
Fully developed flow and temperature fields

\[ Nu_D = 4.364 \quad (8-50) \]

\[ \downarrow \quad q_w = \alpha (t_w - t_B) \]

\[ \text{konst} \quad \text{konst} \]

\[ \Rightarrow t_w - t_B = \text{konst.} \]

If \( t_B \) increases, \( t_w \) must increase as much

Average value including effects of the thermal entrance length

\[
\overline{Nu_D} = \frac{\overline{q_w}}{\overline{\lambda}} = \begin{cases} 
1.953 \left( \frac{1}{Re_D Pr} \frac{x}{D} \right)^{-1/3} \text{ if } \frac{x}{D} \frac{Pr}{Re_D} < 0.03 \\
4.364 + 0.0722 \frac{Re_D Pr}{x/D} \text{ if } \frac{x}{D} \frac{Pr}{Re_D} > 0.03 
\end{cases}
\]
Fully developed flow and temperature fields

\[ \text{Nu}_D = 4.364 \quad (8-50) \]

\[ q_w = \alpha (t_w - t_B) \]

\[ \Rightarrow t_w - t_B = \text{konst.} \]

If \( t_B \) increases, \( t_w \) must increase as much

\( t_w \) highest at the exit!